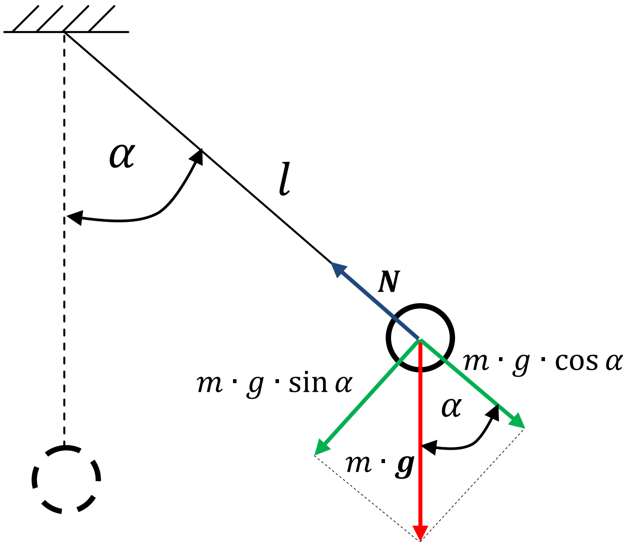
Mathematical pendulum – motion equation

Mathematical pendulum is built with massless rope with length l. One of rope's ends is fixed to the ceiling. To the second end of rope is fixed to the material point with mass m. Material point is then hanged to the rope. When mathematical pendulum is in equilibrium material point is not moving. In equilibrium position gravity force is balanced by rope's tension force. In a certain moment mathematical pendulum was deflected from its equilibrium and was inclined from vertical position by angle α. Mathematical pendulum is under gravity field, so gravity force works on it.



Drawing 1. Mathematical pendulum.

After deflection gravity force divides to two component forces. First component force is parallel to rope and it is balanced by rope's reaction force. Second component force is perpendicular to rope and it is parallel to mathematical pendulum trajectory during motion. Both component forces are proportional to the inclination angle α. In example we assume that motion takes place in vacuum, so there is no air resistance. Material point at the end of pendulum executes harmonic motion on circle.

Motion equation for mathematical pendulum

$$m \cdot \boldsymbol{a} = m \cdot \boldsymbol{g} \cdot \sin\alpha$$

$$I \cdot \varepsilon = -l \cdot m \cdot g \cdot \sin\alpha$$

$$m \cdot l^2 \cdot \frac{d^2\alpha}{dt^2} = -l \cdot m \cdot g \cdot \sin\alpha$$

$$m \cdot l \cdot \frac{d^2\alpha}{dt^2} = -m \cdot g \cdot \sin\alpha$$

$$l \cdot \frac{d^2\alpha}{dt^2} + g \cdot \sin\alpha = 0$$

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l} \cdot \sin\alpha = 0$$

Because for low values of angle α a $\sin\alpha$ is with good approximation equal to angle α ($\sin\alpha \approx \alpha$). We are allowed to make substitution and replace in motion equation $\sin\alpha \to \alpha$.

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l} \cdot \alpha = 0$$

$$\omega^2 = \frac{g}{l}$$

$$\frac{d^2\alpha}{dt^2} + \omega^2 \cdot \alpha = 0$$

$$\alpha(t) = C_1 \cdot e^{i \cdot \omega \cdot t} + C_2 \cdot e^{-i \cdot \omega \cdot t}$$

$$\alpha(t) = \alpha_{max} \cdot \sin\omega \cdot t$$

$$\omega = 2 \cdot \pi \cdot f$$

$$\omega = \frac{2 \cdot \pi}{T}$$

$$\omega^2 = \left(\frac{2 \cdot \pi}{T}\right)^2 = \frac{4 \cdot \pi^2}{T^2}$$

$$T = \sqrt{\frac{4 \cdot \pi^2}{\omega^2}}$$

$$T = 2 \cdot \pi \cdot \sqrt{\frac{1}{\omega^2}}$$

$$T = 2 \cdot \pi \cdot \sqrt{\frac{l}{g}}$$